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Semirelativistic stability of *N*-boson systems bound by $1/r_{ij}$ pair potentials

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Abstract
We analyse a system of self-gravitating identical bosons by means of a semirelativistic Hamiltonian comprising the relativistic kinetic energies of the involved particles and added (instantaneous) Newtonian gravitational pair potentials. With the help of an improved lower bound to the bottom of the spectrum of this Hamiltonian, we are able to enlarge the known region for relativistic stability for such boson systems against gravitational collapse and to sharpen the predictions for their maximum stable mass.

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1. Introduction

In this paper we study the implications of two aspects of relativistic bound systems: the Coulomb (or gravitational) one-body coupling limit, and the effective coupling enhancement induced in a system of many identical particles interacting pairwise. These two effects lead to the conclusion that a system of N identical particles interacting by attractive $1/r$ pair potentials becomes unstable if N is very large. Our principal goal is to sharpen previous bounds on the critical mass of such a system.

Relativistic quantum-mechanical theories imply an upper limit on the strength of the coupling of a single particle bound by an attractive Coulomb potential. Thus for a hydrogen-like one-particle system with mass m, and units such that $\hbar = c = 1$, the upper limits to the allowed coupling v in the potential $-v/r$ are, respectively, $v < 1$ for the Dirac equation and $v < \frac{1}{2}$ for the Klein–Gordon equation. Meanwhile, for the semirelativistic Salpeter equation [1–3] with Hamiltonian $h = \sqrt{p^2 + m^2} - v/r$, Herbst [4] showed that for $v < 2/\pi$

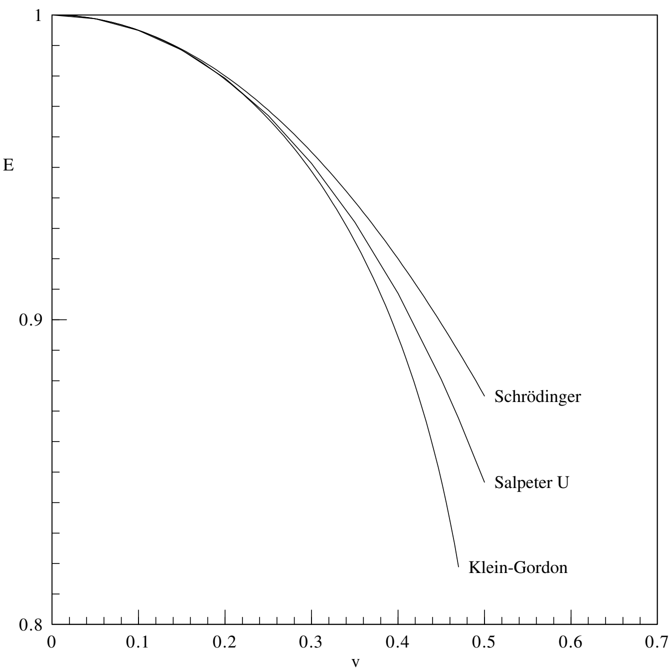

Figure 1. Ground-state energies E (in dimensionless units) for the potential $V(r) = -v/r$ according to the Schrödinger, Salpeter and Klein–Gordon theories. The Salpeter curve is a variational upper bound.

the spectrum of h in $[0, m)$ is discrete and, moreover, he found an explicit lower bound. In summary

$$h = \sqrt{p^2 + m^2} - v/r > m\sqrt{1 - (\pi v/2)^2}, \qquad v < \frac{2}{\pi}. \tag{1}$$

Under the Schrödinger equation, with one or more particles, there is no such coupling restriction; thus the existence of such a coupling limit is essentially a relativistic phenomenon. The Salpeter Hamiltonian has eigenvalues that lie between the corresponding Schrödinger and Klein–Gordon energies. Thus, in addition to exhibiting the relativistic coupling limit, within the allowed couplings, the Salpeter energies are intermediate between those of Schrödinger and Klein–Gordon. For example, for the one-body problem with mass $m = 1$ and the potential $V(r) = -v/r$, the three theories have ground-state eigenvalues that depend on the coupling v as shown in figure 1. The Salpeter result was obtained by the use of a scale-optimized trial function with coordinate expression $\phi(r) = c\,\mathrm{e}^{-r/a}$; it is known analytically that the exact Salpeter curve is bounded below by the Klein–Gordon results for $v < \frac{1}{2}$. A brief review of aspects of Salpeter semirelativistic theory may be found in [5]. If we consider, as we do in this paper, a system of N identical particles interacting pairwise via attractive potentials of the form $-v/r_{ij}$, then the necessary permutation symmetry of the wavefunction effectively enhances the pairwise coupling by a factor of the order of N. This effect, which we shall soon make clear, is most pronounced in the case of bosons.

Particle identity in quantum mechanics is so strong that, in a system of identical particles, the particles lose their individuality; they cannot be separately tracked. This is often helpful for many-particle theory since, when it comes to permutation symmetry, at most two of the many possible Young tableaux need be considered; moreover, many quantities are necessarily equal on the average. We shall now make clear the notion of effective many-body enhancement of the pair couplings which we alluded to above. We do this in the context of the problem that is the main concern of the paper. One of the advantages of the Salpeter semirelativistic theory is that it accommodates a straightforward formulation of the many-body problem. We consider therefore a semirelativistic system of N self-gravitating identical bosons of mass m and momenta $\mathbf{p}_i$, $i = 1, 2, \ldots, N$. This can be described—in a Newtonian approximation, justified to some extent by the assumption of a weak gravitational field—by the Hamiltonian

$$H = \sum_{i=1}^{N} \sqrt{\mathbf{p}_i^2 + m^2} - \sum_{1=i<j}^{N} \frac{\kappa}{r_{ij}}, \qquad \kappa > 0, \tag{2}$$

where, in the Newtonian pair potential, the gravitational interaction strength (determined by the gravitational constant G and the particle mass m) has been encoded in the coupling parameter $\kappa := Gm^2$. The pair distance between the interacting particles i and j is given by $r_{ij} \equiv |\mathbf{x}_i - \mathbf{x}_j|$. If we consider expectations with respect to a normalized boson function Ψ, then we immediately find that there is a relation between H and a scaled two-particle Hamiltonian, namely, $\langle H\rangle = \langle H_2\rangle$, where

$$H_2 := \frac{N}{2}\left[\sqrt{\mathbf{p}_1^2 + m^2} + \sqrt{\mathbf{p}_2^2 + m^2} - (N-1)\frac{\kappa}{r_{12}}\right]. \tag{3}$$

This expectation equality arises because the necessary boson permutation symmetry of the exact N-body ground state Ψ implies that the expectations of the N kinetic-energy terms in H are the same; and similarly for the $\frac{1}{2}N(N-1)$ pair-potential terms. For convenience we have collected these into $N/2$ times a two-body Hamiltonian; we have included the overall $N/2$ factor and written this scaled two-body Hamiltonian as H_2. Thus with respect to the exact wavefunction Ψ we may write $E = \langle H\rangle = \langle H_2\rangle$, where E is the corresponding exact energy. If we denote by E_2 the bottom of the spectrum of the two-boson problem with Hamiltonian H_2, then, since boson symmetry in only two particles is in general a weaker constraint than symmetry in all N particles, it follows that $E \geqslant E_2$. Indeed, the dependence of Ψ on the variables $\{\mathbf{x}_3, \ldots, \mathbf{x}_N\}$ that are not present in H_2 cannot cause $\langle H_2\rangle$ to fall below the bottom of the spectrum of H_2. Thus E_2 provides a lower energy bound to E. We shall sometimes express this as the operator inequality $H \geqslant H_2$. A corresponding upper bound E_g may be found with the aid say of a normalized Gaussian trial function Φ_g: thus $E \leqslant E_g = (\Phi_g, H\Phi_g)$. These energy bounds allow us to compute bounds on the critical mass M_c, the largest allowed mass for such a bound system. In order to make this point clear and to fix ideas, we shall now compute an explicit energy lower bound and from this a lower estimate to M_c. We first have to solve the two-body problem represented by H_2. If we consider for this problem new coordinates $\mathbf{R} = \mathbf{x}_1 + \mathbf{x}_2$, and $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, then the corresponding momenta are related by $\mathbf{p}_1 = \mathbf{p} + \mathbf{P}$ and $\mathbf{p}_2 = \mathbf{p} - \mathbf{P}$. If we introduce a vector $\mathbf{k}$ which is orthogonal to $\mathbf{p}$ and $\mathbf{P}$, then we may consider the following application of the triangle inequality:

$$\begin{aligned} 2(p^2 + m^2)^{\frac{1}{2}} &= |2\mathbf{p} + 2m\mathbf{k}| \\ &= |\mathbf{p} + \mathbf{P} + m\mathbf{k} + \mathbf{p} - \mathbf{P} + m\mathbf{k}| \\ &\leqslant |\mathbf{p}_1 + m\mathbf{k}| + |\mathbf{p}_2 + m\mathbf{k}|. \end{aligned}$$

From this inequality and (3) we conclude the following inequalities

$$H \geqslant H_2 \geqslant N\left[\sqrt{p^2 + m^2} - \frac{(N-1)\kappa}{2r}\right].$$

Consequently, from (1), we have

$$E \geqslant Nm\left[1-\left(\frac{(N-1)\kappa\pi}{4}\right)^2\right] > Nm\left[1-\left(\frac{N\kappa\pi}{4}\right)^2\right].$$

We have replaced $N-1$ by N merely for analytical convenience. Thus we obtain the following N-boson lower energy bound

$$E \geqslant \frac{4m}{\pi\kappa}t(1-t^2)^{\frac{1}{2}}, \qquad t := \frac{N\kappa\pi}{4} \leqslant 1. \tag{4}$$

It turns out that if we maximize the right-hand side of (4) with respect to N, that is to say, with respect to the parameter t, the critical value of t is $\hat{t} = 1/\sqrt{2}$, so that the Herbst coupling inequality is satisfied at the optimal point. Since mass and energy are identified in our units, and $\kappa = Gm^2$, we arrive at the bound $M_c > (2/\pi)/Gm \approx 0.636\,62/Gm$. This detailed calculation shows how an energy bound leads to an estimate for the critical mass M_c. The principal goal of the paper is to refine such estimates. Thus we have here an explicit example of the phenomenon under discussion: if m is the mass of an alpha particle, say, then M cannot be larger than the mass of a modest mountain (we shall present an upper bound to M_c shortly). No such possibility arises from the corresponding nonrelativistic theory.

2. Reduction: 'equivalent' two-body problems

The question of the implications of the necessary permutation symmetry of the states for systems composed of many identical particles is almost as old as quantum mechanics. There are a number of historical threads. Before going into the technical details of our problem, we shall briefly mention two of these. The reasoning leading to the two-particle Hamiltonian H_2 suggests that the energy depends on a reduced density matrix $\rho(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_1', \mathbf{x}_2')$ obtained by integrating

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_N)\Psi(\mathbf{x}_1', \mathbf{x}_2', \mathbf{x}_3, \ldots, \mathbf{x}_N)$$

over all the variables $\mathbf{x}_i$ with $i > 2$. A question raised by this is, what are the necessary features of ρ which characterize it as having come from an N-boson function Ψ? This is called the N-representability problem and goes back at least to the early papers of Löwdin [6] and Coleman [7]: a summary of early work in this direction can be found in the introductory chapters of [8] by Coleman and Rosina. Much of the early work was concerned with atomic and chemical systems. Density-matrix many-body theory and the N-representability problem are still active areas of research [9–11]. We note that, as in the previous paragraph, one can derive energy lower bounds without attempting to solve the N-representability problem generally.

Another story concerns nuclear-type systems, where all the particles enter the motion on an equal footing, and considerations of centre-of-mass motion become important. In order to make this point more explicit, the example of the harmonic oscillator is helpful. We consider briefly the nonrelativistic Hamiltonian given by

$$H_{\mathrm{HO}} = \sum_{i=0}^{N}\frac{\mathbf{p}_i^2}{2m} + \sum_{1=i<j}^{N} v r_{ij}^2. \tag{5}$$

The earliest treatment we know of for this problem is by Houston [12] in 1935; a solution expressed more specifically useful for our purposes was found in 1953 by Post [13]; the solubility of the N-body harmonic-oscillator problem is periodically rediscovered, with justifiable fresh enthusiasm. In units with $\hbar = 1$ the bottom of the spectrum is given exactly by the expression $E_{\rm HO} = 3(N-1)\sqrt{Nv/(2m)}$. The exact ground-state wavefunction is a Gaussian in the $N-1$ orthogonal relative coordinates. If the same reasoning we used to derive the semirelativistic operator bound $H \geqslant H_2$ above is now applied to $H_{\rm HO}$, the resulting lower energy bound obtained is exactly given by $E_L = E_{\rm HO}/\sqrt{2}$. If, instead, the 'reduction' (to a two-body problem) is effected with Jacobi relative coordinates, one obtains a lower bound for the harmonic oscillator equal to $E_{\rm HO}$ itself. This type of reduction has its own history. It is only possible to indicate a few key events of this story in the present short paper. In 1933, just after the discovery of the neutron, physicists began to look at few-nucleon problems. An approach emerged called the 'equivalent two-body method'. It was initiated by Wigner [14] and employed by many researchers [15–20] and eventually found its way into the pages of Rosenfeld's book *Nuclear Forces* [21] in 1948. The idea was always the same, to replace the *N*-body problem by a tractable two-body problem. In many instances the result yielded an energy lower bound, but this was unknown to the workers at the time. The first rigorous results for such problems came in 1956 when Post [22] used Jacobi relative coordinates to show that indeed a lower bound could be constructed. In 1962 Post [23] applied this bound to the gravitational problem with pair potentials of the form $-v/r_{ij}$; together with a Gaussian trial function, the energy was determined to 18%. Rigorous energy bounds with the aid of Jacobi coordinates, and a discussion of the 'equivalent two-body method' may be found in a paper by Hall and Post [24] in 1967. Some similar lower-bound results were later obtained by Levy-Leblond [25] and Stenschke [26]. An independent review and a certain sharpening of results by Hill may be found in [27]. The two streams of activity intersected in a paper by Hall [28] who used a similar argument to that of Coleman [7] to obtain a lower bound for a fermion system in the centre-of-mass frame in terms of a sum over $N-1$ reduced two-particle energies; the bound was also optimized over a certain set of allowed non-orthogonal relative coordinates. Later this lower-bound theory (with non-orthogonal relative coordinates) was extended to the excited states [29]. A variety of alternative lower-bound models and approaches have been developed, for example by Carr [30], Manning [31] and Balbutsev [32]. The nonrelativistic lower bound for the ground-state energy is rediscovered from time to time, for example by Membrado *et al* [33] and Basdevant *et al* [34].

3. Semirelativistic gravitating bosons

We must now return to our main problem, the application of these ideas to a semirelativistic many-body system. The complication that the permutation symmetry (in its spatial aspect) is expressed in the individual-particle coordinates whereas the wavefunction is expressed in relative coordinates remains, and is adjoined by a new difficulty, namely the non-locality of the semirelativistic kinetic-energy operator. Here the *N*-body harmonic-oscillator problem is now no longer solved exactly by the lower bound, but with a finite error less than 0.15% [35].

The Hamiltonian H, among others, has been adopted to investigate spherically symmetric and nonrotating configurations of purely gravitationally interacting bosons forming compact objects known as 'boson stars' [36–40]. This operator H is composed of the relativistically correct expression for the kinetic energy of all the involved bosons and *static* potentials κ/r_{ij} which describe the gravitational forces between these particles. Therefore, it is clearly not possible in this model to take into account retardation effects. In addition, it goes without saying that this approach also omits general-relativistic effects [41–43]. Sufficient conditions have been found both for relativistic stability, which is characterized by the existence of a lower bound on the Hamiltonian H of (2), and for relativistic gravitational collapse, which is inevitable if H is not bounded below. Moreover, semirelativistic bounds have been derived for the maximum possible, or critical, mass M_c of boson stars, that is, the mass beyond which there must be relativistic collapse.

The results of particular interest for this analysis can be summarized as follows. The relativistic kinetic energy $\sqrt{\mathbf{p}^2+m^2}$ satisfies [36] a (tangential [44]) operator inequality, involving an arbitrary real parameter μ with the dimension of mass:

$$\sqrt{\mathbf{p}^2+m^2} \leqslant \frac{\mathbf{p}^2+m^2+\mu^2}{2\mu} \qquad \forall\, \mu > 0.$$

This inequality can be adopted to relate the semirelativistic Hamiltonian H, equation (2), to its nonrelativistic counterpart. A variational bound on the ground-state energy of the nonrelativistic N-particle problem therefore translates into the upper bound $M_c < 1.52/Gm$ [37]. Exploiting the (only numerically computed) nonrelativistic ground-state energy, this bound is refined to $M_c < 1.518/Gm$ [39]. Rewriting H as a sum of one-particle Hamiltonians, each of which is bounded from below by the lowest positive eigenvalue of the Klein–Gordon Schrödinger equation with Coulomb potential, yields a bound[3] to the bottom E of the spectrum of H [38]:

$$E \geqslant Nm\sqrt{\frac{1+\sqrt{1-(N-1)^2\kappa^2}}{2}}, \qquad (N-1)\kappa < 1. \tag{6}$$

The replacement of $N-1$ under the square root by N slightly weakens this bound but allows for its analytic maximization, which entails the analytic lower bound [38]

$$M_c \geqslant \frac{4}{3\sqrt{3}Gm} \simeq \frac{0.7698}{Gm}.$$

Together these estimates constrain M_c to the range $0.7698 < GmM_c < 1.518$. The resulting ratio of upper to lower bounds on M_c is $r_{U/L} \simeq 2.0$. The so-called local-energy theorem may be used to increase the lower bound, whereas a more sophisticated choice of trial functions diminishes the variational upper bound. The combined effect of these improvements is to narrow down the range for M_c to $0.8468 < GmM_c < 1.439$, with upper- to lower-bound ratio of $r_{U/L} \simeq 1.7$ [40].

We have re-analysed the upper bound of [40] with positive non-monotone Hartree wavefunction factors $\phi(r)$. With the factor (before scale optimization) $\phi(r) = c\,e^{-r}(1 + ar)$, $a > 0$, we confirm the findings [40] that the best value of a is about $a \simeq 1$, which yields $GmM_c < 1.438\,71 \approx 1.439$. This ϕ does indeed seem to be close to the best possible Hartree factor. With $a = 1.13$, we get a slight improvement, namely, $GmM_c < 1.438\,54$. With the factor $\phi(r) = c\,e^{-r}(1 - b\,e^{-r})$ we obtain our best result, namely, $GmM_c < 1.43764$ for $b = 0.625$. Thus we have been able to lower the upper bound on the critical mass slightly to $M_c < 1.438/Gm$.

In this paper we tighten the interval allowed for the critical mass of boson stars, by employing an improved *analytic* lower bound [45] on the ground-state energy of the N-particle Hamiltonian (2) for semirelativistic self-gravitating N-boson systems, to a range characterized by an upper- to lower-bound ratio $r_{U/L} \simeq 1.3$. The region of validity of a lower bound on the Hamiltonian H defines the range of relativistic stability of the gravitating N-particle system under study [38]: our improved lower energy bound discussed below increases somewhat the stability region obtained in [38]; for instance, for couplings $\kappa \ll 1$—which allows for large N—this increase of the stability range amounts to an 11% improvement.

[3] [45] discusses this simple '$N/2$ lower energy bound' for arbitrary potentials.

4. Lower bound for self-gravitating semirelativistic N-boson systems

Let $|\Psi\rangle$, $\langle\Psi|\Psi\rangle = 1$, represent the normalized ground state of H, corresponding to its lowest eigenvalue $E \equiv \langle\Psi|H|\Psi\rangle$. Now, the bosonic nature of the identical bound-state constituents forces the eigenstates of H (i.e., their wavefunctions) to be totally symmetric under any permutation of the individual-particle coordinates $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$. The boson permutation symmetry of $|\Psi\rangle$ reduces the N-body problem posed by the Hamiltonian H to a *constrained* two-particle problem [35]:

$$E = \langle\Psi|N\sqrt{\mathbf{p}_N^2 + m^2} - \frac{\gamma\kappa}{r_{N-1,N}}|\Psi\rangle, \qquad \gamma \equiv \frac{N(N-1)}{2}. \tag{7}$$

By use of permutation symmetry, equation (7) may be cast into the equivalent form

$$E = \langle\Psi|\frac{N}{2}\left(\sqrt{\mathbf{p}_1^2 + m^2} + \sqrt{\mathbf{p}_2^2 + m^2}\right) - \frac{\gamma\kappa}{r_{12}}|\Psi\rangle. \tag{8}$$

After removal of the center-of-mass momentum from $\mathbf{p}_1$ and $\mathbf{p}_2$, this apparent two-particle problem reduces to a one-body problem in the relative coordinate and momentum of the particles 1, 2 for which the Klein–Gordon equation with Coulomb interaction gives a lower bound: this eventually yields the bound (6).

The lower bound (6), however, is dramatically improved [45] by the use of Jacobi relative coordinates. The transformation from a given set $\{\mathbf{x}_i, i = 1, 2, \ldots, N\}$ of coordinates to another set $\{\boldsymbol{\rho}_k, k = 1, 2, \ldots, N\}$ may be defined by a matrix $B = (B_{ki})$: $\boldsymbol{\rho} = B\mathbf{x}$. The orthogonality $B^{-1} = B^{\mathrm{T}}$ of B is not mandatory but may prove to be convenient. The momenta $\{\boldsymbol{\pi}_i\}$ conjugate to the $\{\boldsymbol{\rho}_i\}$ are then also determined by $\boldsymbol{\pi} = (B^{-1})^{\mathrm{T}}\mathbf{p} = B\mathbf{p}$. The transformation to Jacobi relative coordinates is represented by an orthogonal matrix with the first row given by

$$B_{1i} = \frac{1}{\sqrt{N}} \qquad \forall\, i = 1, 2, \ldots, N,$$

whereas, for all $2 \leqslant k \leqslant N$, in the kth row only the first k entries are nonzero:

$$B_{ki} = \frac{1}{\sqrt{k(k-1)}} \quad \forall\, i = 1, 2, \ldots, k-1, \qquad B_{kk} = -\sqrt{\frac{k-1}{k}},$$
$$B_{ki} = 0 \qquad \forall\, i = k+1, k+2, \ldots, N.$$

Evidently, by the definition of B its first row generates the usual center-of-mass variable $\boldsymbol{\rho}_1$, while its second row introduces a pair distance $\boldsymbol{\rho}_2 = (\mathbf{x}_1 - \mathbf{x}_2)/\sqrt{2}$.

Any boson state $|\Phi\rangle$ is symmetric under permutations of all *individual-particle* coordinates. However, a non-Gaussian boson state is not necessarily symmetric in the Jacobi *relative* coordinates. Nevertheless, as has been shown in appendix A of [45], each such $|\Phi\rangle$ satisfies, for all $i, k \geqslant 2$, the N-representability identities

$$\langle\Phi|\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_k|\Phi\rangle = \delta_{ik}\langle\Phi|\boldsymbol{\rho}_2^2|\Phi\rangle, \qquad \langle\Phi|\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_k|\Phi\rangle = \delta_{ik}\langle\Phi|\boldsymbol{\pi}_2^2|\Phi\rangle. \tag{9}$$

Now, for the sake of notational simplicity, let us introduce some abbreviations:

$$\lambda \equiv \frac{N-1}{N}, \qquad a \equiv \frac{1}{\sqrt{\lambda}} = \sqrt{\frac{N}{N-1}}, \qquad b \equiv \sqrt{\frac{N-2}{N-1}}, \qquad c \equiv \frac{b}{a} = \sqrt{\frac{N-2}{N}}.$$

These parameters λ, a, b, c are, of course, related by $a^2 + b^2 = 2$ and $1 + c^2 = 2\lambda$. In terms of Jacobi relative coordinates, the expectation value (7) then becomes

$$E = \langle\Psi|N\sqrt{(a\boldsymbol{\pi}_1 - \sqrt{\lambda}\boldsymbol{\pi}_N)^2 + m^2} - \frac{\gamma\kappa}{|a\boldsymbol{\rho}_N - b\boldsymbol{\rho}_{N-1}|}|\Psi\rangle.$$

We assume that the eigenstate $|\Psi\rangle$ depends on $\{\boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \ldots, \boldsymbol{\rho}_N\}$ but not on $\boldsymbol{\rho}_1$. A lemma shown in [46] allows us to remove the center-of-mass momentum $\boldsymbol{\pi}_1$ from the kinetic term. Thus the N-body ground-state energy E simplifies to

$$E = \langle\Psi|N\sqrt{\lambda\boldsymbol{\pi}_N^2 + m^2} - \frac{\gamma\kappa}{|a\boldsymbol{\rho}_N - b\boldsymbol{\rho}_{N-1}|}|\Psi\rangle. \tag{10}$$

Focusing on the $(N-1, N)$ subsystem we introduce new coordinates $\{\mathbf{R}, \mathbf{r}\}$ and their conjugate momenta $\{\mathbf{P}, \mathbf{p}\}$, by performing the coordinate transformation

$$\begin{pmatrix}\mathbf{R}\\ \mathbf{r}\end{pmatrix} = O\begin{pmatrix}\boldsymbol{\rho}_N\\ \boldsymbol{\rho}_{N-1}\end{pmatrix}, \qquad \begin{pmatrix}\mathbf{P}\\ \mathbf{p}\end{pmatrix} = \frac{O}{2}\begin{pmatrix}\boldsymbol{\pi}_N\\ \boldsymbol{\pi}_{N-1}\end{pmatrix}.$$

The expectation value (10) suggests the most favourable choice of the matrix O:

$$O \equiv \begin{pmatrix}b & a\\ a & -b\end{pmatrix} = O^{\mathrm{T}}, \qquad O^{\mathrm{T}}O = O^2 = 2.$$

Upon this change of variables, the ground-state energy E of the Hamiltonian H is given by the expectation value $E = \langle\Psi|\mathcal{H}|\Psi\rangle$ of the two-particle Hamiltonian

$$\mathcal{H} \equiv N\sqrt{(\mathbf{p} + c\mathbf{P})^2 + m^2} - \frac{\gamma\kappa}{r}, \qquad r \equiv |\mathbf{r}|.$$

It may be proved that $\mathcal{H}$ is bounded from below by the Hamiltonian entering in the expectation value on the right-hand side of equation (8) (see appendix B of [45]).

For the new momenta $\mathbf{P}$ and $\mathbf{p}$, the identities (9) translate into the constraints

$$\langle\Phi|\mathbf{P}^2|\Phi\rangle = \langle\Phi|\mathbf{p}^2|\Phi\rangle \qquad \text{and} \qquad \langle\Phi|\mathbf{P}\cdot\mathbf{p}|\Phi\rangle = 0.$$

Consequently, we have to look for the bottom of the spectrum of the constrained problem posed by the operator $\mathcal{H}$ in a domain $\mathcal{D}$ restricted by these conditions:

$$\mathcal{D} = \{|\varphi\rangle \in L^2(\Re^6) : \langle\varphi|\mathbf{P}^2|\varphi\rangle = \langle\varphi|\mathbf{p}^2|\varphi\rangle, \langle\varphi|\mathbf{P}\cdot\mathbf{p}|\varphi\rangle = 0\}.$$

This bottom $\mathcal{E}$ of the spectrum of $\mathcal{H}$, of course, provides a lower bound to E:

$$E = \langle\Psi|\mathcal{H}|\Psi\rangle \geqslant \inf_{\substack{|\varphi\rangle\in\mathcal{D}\\ \langle\varphi|\varphi\rangle=1}} \langle\varphi|\mathcal{H}|\varphi\rangle \equiv \mathcal{E}.$$

Let $|\psi\rangle \in \mathcal{D}$, $\langle\psi|\psi\rangle = 1$, be the eigenstate of the Hamiltonian $\mathcal{H}$ corresponding to this lowest eigenvalue $\mathcal{E}$. The eigenvalue equation of $\mathcal{H}$ satisfied by $|\psi\rangle$ reads

$$N\sqrt{(\mathbf{p} + c\mathbf{P})^2 + m^2}|\psi\rangle = \left(\mathcal{E} + \frac{\gamma\kappa}{r}\right)|\psi\rangle.$$

By squaring this relation and remembering the constraints that define $\mathcal{D}$ we get

$$\mathcal{E}^2 - N^2m^2 = 4\gamma\langle\psi|\mathbf{p}^2 - \frac{\kappa\mathcal{E}}{2r} - \frac{\gamma\kappa^2}{4r^2}|\psi\rangle. \tag{11}$$

Now, by assumption, $|\psi\rangle$ is the lowest eigenstate of $\mathcal{H}$ but not necessarily of the one-particle Kratzer-type [47] operator in equation (11). According to the variational principle, the (well-known) lowest eigenvalue of this Kratzer-type Hamiltonian provides a lower bound on the expectation value in equation (11). Solving the implicit inequality for $\mathcal{E}$ yields a lower bound to $\mathcal{E}$, and thus to E [45]; this lower bound is nothing but the lowest positive eigenvalue of the corresponding Klein–Gordon Schrödinger equation [48] for gravitational interaction of appropriate strength:

$$E \geqslant Nm\sqrt{\frac{1 + \sqrt{1 - \gamma\kappa^2}}{2}}, \qquad \gamma\kappa^2 < 1. \tag{12}$$

Our improved lower bound (12) on the ground-state energy of any self-gravitating N-boson system is of the same form as the bound (6) but with $\gamma \equiv N(N-1)/2$ replacing $(N-1)^2$, which is favourable since $N(N-1)/2 < (N-1)^2$ for $N > 2$.

5. Semirelativistic stability and critical mass of boson stars

Let us now analyse the implications of the improved lower energy bound (12) for both stability against gravitational collapse and maximum mass of boson stars.

The existence of a lower bound on the spectrum of the Hamiltonian operator H guarantees the stability of the self-gravitating boson system against relativistic gravitational collapse. The region of validity of such kind of lower energy bound delimits the stability range of the bound state described by H. By construction, our bound (12) holds for all N satisfying $N(N-1)\kappa^2 < 2$. This stability region is larger than the one, $(N-1)\kappa < 4/\pi$, found in [38]. For large values of N, allowed for sufficiently small couplings κ, this gain amounts to $\pi/2\sqrt{2} = 1.11$. In terms of Newton's constant G and the particle mass m, a *sufficient condition for relativistic stability* thus is that the particle number N fulfils the constraint

$$N(N-1) < \frac{2}{(Gm^2)^2}.$$

Following [38], in order to allow for a discussion by elementary methods, we weaken equation (12) by replacing the exact N dependence $\gamma \equiv N(N-1)/2$ by $N^2/2$:

$$E \geqslant Nm\sqrt{\frac{1+\sqrt{1-N^2\kappa^2/2}}{2}}, \qquad N\kappa < \sqrt{2}.$$

Evidently, even this weakened lower bound is still above the lower bound (6) for all $N > 2+\sqrt{2} \simeq 3.41$; for large N, the weaker bound approaches the exact one. The (single) maximum of this weakened lower bound is situated at the critical point $\hat{N} = 4/3\kappa$, which is, fortunately, in the interior of the region of validity of our lower bound (12) on the Hamiltonian H as $\hat{N} < \sqrt{2}/\kappa$. This maximum thus constitutes the (improved) lower bound on the critical mass M_c of boson stars

$$M_c \geqslant \frac{4\sqrt{2}m}{3\sqrt{3}\kappa} = \frac{4\sqrt{2}}{3\sqrt{3}Gm} = \frac{1.088\,66}{Gm}. \tag{13}$$

This lower M_c bound is larger by exactly a factor $\sqrt{2}$ than the result of [38]. Combining the lower bound (13) with the Rayleigh–Ritz upper bound of [40] tightens the (Newtonian-limit) prediction for M_c to $1.088\,66 < GmM_c < 1.439$, reducing thus the ratio between upper and lower bounds on M_c to $r_{U/L} \simeq 1.3$.

In summary, with the aid of an improved lower bound [45] (based on the relative coordinates of the bound-state constituents) on the bottom of the spectrum of the semirelativistic N-boson Hamiltonian (2) with gravitational interaction we have succeeded in enlarging the range of semirelativistic stability of boson stars and in halving the theoretical uncertainty in the maximum mass of boson stars.

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